Transport theory yields renormalization-group equations

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We show that dissipative transport and renormalization can be described in a single theoretical framework. The appropriate mathematical tool is the Nakajima-Zwanzig projection technique. We illustrate our result in the case of interacting quantum gases, where we use the Nakajima-Zwanzig approach to investigate the renormalization-group flow of the effective two-body interaction. [S1063-651X(97)02205-8]

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I. INTRODUCTION

The basic theme of statistical mechanics—how to obtain a system's macroscopic properties from the laws of its underlying microscopic dynamics—appears in many variations. Two out of many examples are the problem of determining critical exponents at second-order phase transitions and the problem of deriving macroscopic transport equations. The former is usually tackled with the help of Wilson's renormalization group [1–4], a mathematical tool that allows one to iteratively eliminate short-wavelength modes and thus to arrive at effective (''renormalized'') theories that describe the dynamics on successively larger length scales. The latter has been tackled in various ways, among them the so-called projection technique by Nakajima [5], Zwanzig [6], Mori [7], and Robertson [8]. Eliminating unmonitored, rapidly oscillating degrees of freedom from the equation of motion by means of suitable projections in the space of observables, the projection technique yields closed (but generally no longer Markovian) ''transport equations'' for the selected macroscopic degrees of freedom.

While the two methods—Wilson's renormalization group and Zwanzig's projection technique—may appear quite different in their mathematical formulation, they are very similar in spirit. In both cases one strives to focus on selected features of the dynamics (its infrared limit or the evolution of only few macroscopic observables) deemed interesting and to this end devises a systematic procedure for eliminating all other, ''irrelevant'' degrees of freedom (a procedure commonly referred to as ''coarse graining''). Discarding thus unnecessary baggage from the problem at hand, one succeeds in describing the interesting features of the dynamics without ever having to solve the complete, and far too complicated, microscopic theory. This similarity of the basic approach suggests that renormalization and the transition from microdynamics to macroscopic transport are in fact closely related procedures and that it should be possible to cast them into a common theoretical framework.

Building a bridge between renormalization and transport theory would not only be satisfying conceptually, but would also help tackle a variety of practical problems. Often the macroscopic evolution of a complex quantum system exhibits *both* dissipation *and* modified, renormalized dynamical parameters such as effective masses or effective interactions. Let us consider, for example, liquid ^{3}He or nuclear matter away from equilibrium. In order to formulate a macroscopic transport theory for such an interacting fermion system one must perform *two* consecutive coarse-graining procedures: first eliminating short-wavelength modes to arrive at an effective (renormalized) theory for quasiparticle excitations close to the Fermi surface, which typically feature effective masses and screened interactions [9–11], and then discarding their statistical correlations to obtain an Uehling-Uhlenbeck-type transport equation for the single-quasiparticle distribution. The description of the macroscopic dynamics therefore requires an appropriate combination of renormalization and statistical coarse graining.

Clearly, the two coarse grainings do not commute; the latter (statistical coarse graining) is contingent upon the former (renormalization). For instance, the renormalization-group flow yields screening [11] and hence renders the interaction range finite, thus generating that separation of scales which is indispensable for the subsequent derivation of a Markovian transport theory [12]. But what happens if scales converge rather than separate upon renormalization? How then are renormalization and statistical coarse graining best combined? More generally, what is the connection between effective dynamics and dissipation? Does their interplay lead to interesting new phenomena? To what extent can renormalization-group techniques be applied to study nonequilibrium, dissipative processes? How do transport coefficients change under renormalization-group transformations? Is it always true that transport coefficients are renormalized by simply trading bare masses and couplings for their renormalized counterparts, while keeping the form of the functional dependence on these parameters [13]? In addition, somewhat speculative, are there ''universality classes'' of transport theories? These and other issues might be best approached in a unified mathematical framework that encompasses both renormalization and dissipative transport as special cases.

There has already been some progress towards such a unified picture. The success of Anderson's ''poor man's scaling'' approach to the Kondo problem [14], Seke's projection-method treatment of the nonrelativistic Lamb shift [15], the calculation of the one-loop renormalization of ϕ^4 theory by means of Bloch-Feshbach techniques [16], and a recent renormalization-group study of interacting fermion systems within a purely algebraic framework [17] suggest that one can formulate Wilson's renormalization in terms of projections in Hilbert space, completely analogous to the projections in the space of observables, which, in the

Nakajima-Zwanzig approach, lead to macroscopic transport equations.

In the present paper I wish to make this analogy even more explicit. I will show that one can actually obtain renormalization-group equations within the Nakajima-Zwanzig projection approach and that hence renormalization can be embedded into the general mathematical framework of transport theory. After a brief introduction to the projection technique (Sec. II) and a discussion of various approximations (Sec. III) I shall isolate the dissipative and nondissipative parts of the macroscopic dynamics and show that the latter is governed by an effective, renormalized Hamiltonian (Sec. IV). For illustration, these general ideas are then applied to studying the low-energy dynamics of interacting quantum (Bose and Fermi) gases, in particular the renormalization-group flow of their effective two-body interaction (Secs. V and VI). Finally, I shall conclude with a brief summary in Sec. VII.

II. PROJECTION TECHNIQUE

In this section I give a very brief introduction to the Nakajima-Zwanzig projection technique [5–8]. More details can be found in several textbooks [18] and in recent reviews [12,19].

When studying the dynamics of a macroscopic quantum system away from equilibrium, one typically monitors the evolution of the expectation values

$$g_a(t) := \mathrm{tr}[\rho(t) G_a] \tag{1}$$

of only a very small set of selected (''relevant'') observables $\{G_a\}$. These evolve according to

$$\dot{g}_a(t) = i(\rho(t)|\mathcal{L}G_a), \tag{2}$$

with $\rho(t)$ being the statistical operator, $\mathcal{L}$ the Liouvillian

$$\mathcal{L} := \hbar^{-1}[H,*] \tag{3}$$

associated with the Hamiltonian H, and the inner product $(\,|\,)$ defined as

$$(A|B) := \mathrm{tr}[A^\dagger B]. \tag{4}$$

The equation of motion in the form (2) does not constitute a closed system of differential equations for the selected expectation values $\{g_a(t)\}$; its right-hand side will generally depend not just on the selected, but also on all the other ''irrelevant'' degrees of freedom. With the help of the projection technique to be sketched below, these irrelevant degrees of freedom can be systematically eliminated from the equation of motion, in exchange for non-Markovian and (possibly) nonlinear features of the resulting closed ''transport equation'' for the $\{g_a(t)\}$. Mapping thus the influence of irrelevant degrees of freedom onto, among other features, a nonlocal behavior in time opens the way to the exploitation of well-separated time scales and hence serves as a good starting point for powerful approximations such as the Markovian and quasistationary limits. Indeed, in this fashion one can derive many of the well-known equations of nonequilibrium statistical mechanics, for example, rate, quantum Boltzmann, Master, Langevin-Mori, and even time-dependent Hartree-Fock equations. As I will show later in this paper, the same projection technique allows one to derive renormalization-group equations.

The projection technique is based on a clever insertion of projection operators into the equation of motion (2). A projection operator is any operator $\mathcal{P}$ that satisfies $\mathcal{P}^2=\mathcal{P}$; its complement, which is also a projection operator, is denoted by $\mathcal{Q} := 1-\mathcal{P}$. The projection operators, like the Liouvillian, are so-called superoperators: they do not act in Hilbert space but in the space of observables (Liouville space). For our purposes we consider projectors that project arbitrary vectors in Liouville space onto the subspace spanned by the unit operator and by the relevant observables $\{G_a\}$, i.e., for which

$$\mathcal{P}A = A \;\Leftrightarrow\; A \in \mathrm{span}\{1, G_a\}. \tag{5}$$

For simplicity we assume that the Hamiltonian and hence the Liouvillian, as well as the relevant observables, are not explicitly time dependent. In contrast, we allow the projector to depend on the expectation values $\{g_a(t)\}$ of the relevant observables, thus making it an implicit function of time: $\mathcal{P}(t) \equiv \mathcal{P}[g_a(t)]$, with the sole restriction that for any observable A,

$$\left(\rho(t)\left|\frac{d}{dt}\mathcal{P}(t)\;A\right.\right) = 0. \tag{6}$$

For the time being we admit any projector that satisfies the two constraints (5) and (6). Later, in Sec. IV B, we shall make a specific choice for $\mathcal{P}(t)$.

Now let $\mathcal{T}(t',t)$ be the (super)operator defined by the differential equation

$$\frac{\partial}{\partial t'}\mathcal{T}(t',t) = -i\;\mathcal{Q}(t')\mathcal{L}\mathcal{Q}(t')\mathcal{T}(t',t), \tag{7}$$

with the initial condition $\mathcal{T}(t,t)=1$. It may be pictured as describing the evolution of the system's *irrelevant* degrees of freedom. With its help the equation of motion for the selected expectation values $\{g_a(t)\}$ can be cast into the—still exact—form

$$\begin{aligned}\dot{g}_a(t) = {} & i(\rho(t)|\mathcal{P}(t)\mathcal{L}G_a) \\ & - \int_0^t dt'\;(\rho(t')|\mathcal{P}(t')\mathcal{L}\mathcal{Q}(t')\mathcal{T}(t',t)\mathcal{Q}(t)\mathcal{L}G_a) \\ & + i(\rho(0)|\mathcal{Q}(0)\mathcal{T}(0,t)\mathcal{Q}(t)\mathcal{L}G_a)\end{aligned} \tag{8}$$

for any time $t \geqslant 0$. Comparing this form of the equation of motion with the original form (2) we notice that, apart from the replacement $(\rho| \to (\rho|\mathcal{P}$, there are two additional terms: (i) an integral (''memory'') term, containing contributions from all times between the initial and the present time, and (ii) a ''residual force'' term describing the effect of irrelevant components in the initial state. The physical meaning of both terms can be easily discerned if read from left to right. (i) At time $t' < t$ relevant degrees of freedom (projected out by $\mathcal{P}$) couple via an interaction ($\mathcal{L}$) to irrelevant degrees of freedom (projected out by $\mathcal{Q}$), which subsequently evolve in time ($\mathcal{T}$) and, due to a second interaction ($\mathcal{L}$), acquire relevancy again, thus influencing the evolution

of the relevant observable G_a at the present time t. (ii) Irrelevant components in the initial state ($\mathcal{Q}$) evolve in time ($\mathcal{T}$) and, due to interaction ($\mathcal{L}$), acquire relevancy at the present time t.

In many practical applications the irrelevant component of the initial state can be shown to vanish, or else to be negligible, in which case the residual force term can be dropped from the equation of motion. What remains then is the desired closed system of (possibly nonlinear) coupled integro-differential equations for the selected expectation values $\{g_a(t)\}$. The principal feature of these closed equations is that they are non-Markovian: Future expectation values of the selected observables are predicted not just on the basis of their present values, but based on their entire history.

III. APPROXIMATIONS

A. Second-order perturbation theory

Often the Liouvillian can be split into a free part and an interaction part,

$$\mathcal{L}=\mathcal{L}^{(0)}+\mathcal{V}, \tag{9}$$

corresponding to a decomposition $H=H^{(0)}+V$ of the Hamiltonian. Provided free evolution does not mix relevant and irrelevant degrees of freedom, i.e., provided

$$[\mathcal{L}^{(0)},\mathcal{P}(t)]=0, \tag{10}$$

then in the memory term $\mathcal{PLQ}=\mathcal{PVQ}$ and $\mathcal{QLP}=\mathcal{QVP}$; hence the memory term is at least of second order in the interaction. To second order, therefore, one can simply replace

$$\mathcal{Q}(t')\mathcal{T}(t',t)\mathcal{Q}(t)\ \rightarrow\ \mathcal{Q}(t')\mathcal{T}^{(0)}(t',t)\mathcal{Q}(t), \tag{11}$$

where $\mathcal{T}^{(0)}$ is a time-ordered exponential of $\mathcal{QL}^{(0)}\mathcal{Q}$. Using Eq. (10) and $\mathcal{Q}(t_2)\mathcal{Q}(t_1)=\mathcal{Q}(t_2)$, all the $\mathcal{Q}$'s appearing in $\mathcal{T}^{(0)}$ can be shuffled to the left and absorbed into $\mathcal{Q}(t')$, allowing one to replace further

$$\mathcal{T}^{(0)}(t',t)\ \rightarrow\ \mathcal{U}^{(0)}(t',t):=\exp[i(t-t')\mathcal{L}^{(0)}]. \tag{12}$$

One thus obtains (without residual force)

$$\dot{g}_a(t)=i(\rho(t)|\mathcal{P}(t)\mathcal{L}G_a) - \int_0^t d\tau\ (\rho(t-\tau)|\mathcal{P}(t-\tau)\mathcal{V}\mathcal{Q}(t-\tau)\mathcal{U}^{(0)}(0,\tau)\mathcal{V}G_a)\ . \tag{13}$$

B. Markovian limit

We have seen that predictions of future expectation values of the selected observables generally depend in a complicated manner on both their present expectation values and their past history. There are thus two distinct time scales: (i) the scale $\tau_{\rm rel}$, or several scales $\{\tau_{\rm rel}^i\}$, on which the selected expectation values $\{g_a(t)\}$ evolve and (ii) the *memory time* $\tau_{\rm mem}$ which characterizes the length of the time interval that contributes significantly to the memory integral. Loosely speaking, the memory time determines how far back into the past one has to reach in order to make predictions for the further evolution of the selected observables. If this memory time is small compared to the typical time scale on which the selected observables evolve, $\tau_{\rm mem}\ll\tau_{\rm rel}$, then memory effects can be neglected and predictions for the selected observables can be based solely on their present values. One may then assume that in the memory term $g_a(t')\approx g_a(t)$ and hence replace

$$\mathcal{P}[g_a(t')]\rightarrow\mathcal{P}[g_a(t)],$$
$$(\rho(t')|\mathcal{P}(t')\rightarrow(\rho(t)|\mathcal{P}(t). \tag{14}$$

This is the Markovian limit.

Closely related to the Markovian limit is the quasistationary limit: At times $t\gg\tau_{\rm mem}$ it no longer matters for the dynamics when exactly the evolution started, and hence in Eq. (8) the integration over the system's history may just as well extend from $-\infty$ to t rather than from 0 to t. In the Markovian and quasistationary limits the equation of motion, to second order and without residual force, simplifies to

$$\dot{g}_a(t)=i(\rho(t)|\mathcal{P}(t)\mathcal{L}G_a) - \int_0^\infty d\tau\ (\rho(t)|\mathcal{P}(t)\mathcal{V}\mathcal{Q}(t)\mathcal{U}^{(0)}(0,\tau)\mathcal{V}G_a)\ . \tag{15}$$

This approximate transport equation for the $\{g_a(t)\}$ shall be the basis of our further investigations.

IV. ANALYSIS

A. Reformulation of the transport equation

Starting from the approximate transport equation (15), we eventually wish to discern two of the main features of macroscopic transport: (i) dissipation and (ii) the modification (''renormalization'') of the effective interaction. The latter will then, in Sec. V, lead on to the consideration of renormalization-group equations.

To this end we first split the free evolution operator into its symmetric and antisymmetric parts

$$\mathcal{U}^{(0)}(0,\tau)=\tfrac{1}{2}[\mathcal{U}^{(0)}(0,\tau)+\mathcal{U}^{(0)}(\tau,0)] + \tfrac{1}{2}[\mathcal{U}^{(0)}(0,\tau)-\mathcal{U}^{(0)}(\tau,0)], \tag{16}$$

use

$$\int_0^\infty d\tau\ [\mathcal{U}^{(0)}(0,\tau)+\mathcal{U}^{(0)}(\tau,0)]=2\pi\delta(\mathcal{L}^{(0)}) \tag{17}$$

in the symmetric part, and in the antisymmetric part exploit the liberty (thanks to the Markovian limit) to insert free evolution operators $\mathcal{U}^{(0)}(0,\tau)$ and $\mathcal{U}^{(0)}(\tau,0)$ in front of $\mathcal{P}(t)$ or G_a, respectively. We thus obtain

$$\dot{g}_a(t)=i(\rho(t)|\mathcal{P}(t)\mathcal{L}_{\rm eff}(t)G_a) - \pi(\rho(t)|\mathcal{P}(t)\mathcal{V}\mathcal{Q}(t)\delta(\mathcal{L}^{(0)})\mathcal{V}G_a), \tag{18}$$

where

$$\mathcal{L}_{\rm eff}(t)=\mathcal{L}+\delta\mathcal{L}(t) \tag{19}$$

denotes a (possibly time-dependent) ''effective'' Liouvillian determined by

$$\delta\mathcal{L}(t)=-\frac{i}{2}\int_0^\infty d\tau\{[\mathcal{V}(\tau),\mathcal{V}]-[\mathcal{P}(t)\mathcal{V}(\tau)\mathcal{P}(t),\mathcal{P}(t)\mathcal{V}\mathcal{P}(t)]\} \tag{20}$$

and

$$\mathcal{V}(\tau)=\hbar^{-1}[V(\tau),*] \tag{21}$$

is the commutator with the interaction-picture operator

$$V(\tau):=\mathcal{U}^{(0)}(0,\tau)V. \tag{22}$$

Next we evaluate the reformulated transport equation (18) by making a suitable choice for the yet undetermined projector $\mathcal{P}(t)$.

B. Robertson projector

Associated with the expectation values (1) of the selected observables is, at each time t, a generalized canonical state

$$\rho_{\rm rel}(t):=Z(t)^{-1}\exp[-\lambda^a(t)G_a], \tag{23}$$

with the partition function

$$Z(t):=\mathrm{tr}\exp[-\lambda^a(t)G_a] \tag{24}$$

and the Lagrange parameters $\{\lambda^a(t)\}$ adjusted such as to satisfy the constraints

$$\mathrm{tr}[\rho_{\rm rel}(t)G_a]=g_a(t). \tag{25}$$

We have used Einstein's convention: Repeated upper and lower indices are to be summed over. The generalized canonical state $\rho_{\rm rel}(t)$, among all states that satisfy the constraints (25), is the one that maximizes the von Neumann entropy

$$S[\rho]:=-k\ \mathrm{tr}\ (\rho\ln\rho). \tag{26}$$

For this reason it may be considered the ''least biased'' or ''maximally noncommittal'' with regard to the unmonitored degrees of freedom; it is sometimes called the ''relevant part'' of the full statistical operator $\rho(t)$.

There exists a unique time-dependent projector $\mathcal{P}_R(t)$ that projects arbitrary vectors in Liouville space onto the subspace spanned by the unit operator and by the relevant observables $\{G_a\}$, the projection being orthogonal with respect to the time-dependent scalar product

$$\langle A;B\rangle^{(t)}:=\int_0^1 d\mu\ \mathrm{tr}[\rho_{\rm rel}(t)^\mu A^\dagger\rho_{\rm rel}(t)^{1-\mu}B]. \tag{27}$$

This projector $\mathcal{P}_R(t)$ satisfies both conditions (5) and (6) and, moreover, can be shown to yield

$$(\rho(t)|\mathcal{P}_R(t)=(\rho_{\rm rel}(t)| \tag{28}$$

at all times. This special choice for $\mathcal{P}(t)$, originally proposed by Robertson [8,20], has the important advantage that in contrast to other frequently used projectors such as the Mori projector [7], it permits the derivation of closed transport equations valid arbitrarily far from equilibrium. We shall use this projector throughout the remainder of the paper (and for brevity, we shall immediately drop the subscript R).

With the Robertson projector the transport equation (18) takes the form

$$\dot g_a(t)=i(\rho_{\rm rel}(t)|\mathcal{L}_{\rm eff}(t)G_a)+M_{ca}(t)\lambda^c(t), \tag{29}$$

where

$$M_{ca}(t):=\pi\langle\mathcal{Q}(t)\mathcal{V}G_c;\delta(\mathcal{L}^{(0)})\mathcal{Q}(t)\mathcal{V}G_a\rangle^{(t)} \tag{30}$$

is a matrix whose eigenvalues are all real and non-negative. In this formulation it is particularly easy to distinguish the dissipative and nondissipative parts of the dynamics: As we will show, the second term in Eq. (29) is solely responsible for dissipation, whereas the first term yields nondissipative dynamics governed by a modified (renormalized) effective Hamiltonian.

C. Dissipation

An observer who monitors only the selected degrees of freedom does not have complete information about the system's microstate. A suitable measure for this lack of information is the entropy $S[\rho_{\rm rel}(t)]$ associated with the relevant part of the statistical operator. It is sometimes called the ''relevant entropy.'' This relevant entropy generally varies in time: It changes at a rate

$$\dot S[\rho_{\rm rel}(t)]=k\ \lambda^a(t)\dot g_a(t). \tag{31}$$

Within our approximations—perturbation theory and Markovian limit—this rate may be evaluated by inserting the transport equation (29). Its first term does not contribute to the change of relevant entropy; only its second term yields a nontrivial contribution

$$\dot S[\rho_{\rm rel}(t)]=k\ M_{ca}(t)\lambda^c(t)\lambda^a(t)\geqslant 0. \tag{32}$$

The relevant entropy thus increases monotonically, reflecting dissipation and irreversibility of the macroscopic dynamics. It stays constant if and only if the second term in Eq. (29) vanishes (the ''adiabatic limit'').

Of course, our finding represents one particular case of the more general H theorem. That the relevant entropy can never decrease is a direct consequence of the Markovian limit and hence holds true whenever the system exhibits a clear separation of time scales [12].

D. Effective Hamiltonian

The nondissipative part of the macroscopic dynamics is encoded entirely in the first term of Eq. (29). So in the adiabatic limit, which we shall consider from now on, the transport equation simplifies to

$$\dot g_a^{\rm ad}(t)=i(\rho_{\rm rel}(t)|\mathcal{L}_{\rm eff}(t)G_a). \tag{33}$$

This is very similar to the original equation of motion (2), yet with ρ replaced by $\rho_{\rm rel}$ and $\mathcal{L}$ replaced by $\mathcal{L}_{\rm eff}$.

Here we make a special choice for the selected observables, one that will directly lead to the renormalization group. We presume that we are interested in features of the macroscopic system (for example, its long-wavelength properties) that can be represented by observables acting merely in some subspace of the original Hilbert space (e.g., in the subspace spanned by all many-particle states with momenta below a given cutoff). Let the operator that projects the original Hilbert space onto this selected subspace be denoted by P and its complement by $Q=1-P$. Selected observables are then all those of the form PAP, with A being an arbitrary Hermitian operator. This choice of relevant observables gives rise to a particularly simple representation of the Robertson projector,

$$\mathcal{P}A = PAP + \frac{\mathrm{tr}(QA)}{\mathrm{tr}Q}Q \quad \forall \ A, \tag{34}$$

which no longer varies in time.

We now decompose the microscopic Hamiltonian H into a nonmixing ''free'' part

$$H^{(0)} = PHP + QHQ \tag{35}$$

and a mixing ''interaction''

$$V = PHQ + QHP \tag{36}$$

and split the Liouvillian correspondingly. Since

$$[\mathcal{L}^{(0)}, \mathcal{P}] = 0, \tag{37}$$

this decomposition is suitable for perturbation theory. With the property

$$\mathcal{P}\mathcal{V}\mathcal{P} = 0 \tag{38}$$

we can immediately evaluate Eq. (20) to obtain

$$\delta\mathcal{L} = \hbar^{-1}[\delta H, *], \tag{39}$$

where δH is given by

$$\delta H = \frac{i}{2\hbar}\int_0^\infty d\tau\, [V, V(\tau)]$$

$$= -\frac{1}{2\hbar}\left[V, \frac{1}{\mathcal{L}^{(0)}}V\right] + i\,\frac{\pi}{2\hbar}[V, \delta(\mathcal{L}^{(0)})V]. \tag{40}$$

This yields then

$$\mathcal{P}\mathcal{L}_{\mathrm{eff}}\mathcal{P}A = \hbar^{-1}[H_{\mathrm{eff}}, PAP] \quad \forall \ A, \tag{41}$$

where we have identified the effective Hamiltonian

$$H_{\mathrm{eff}} = PHP + \Sigma, \tag{42}$$

which is not just the projection PHP of the original Hamiltonian, but contains an extra term

$$\Sigma = -\frac{1}{2\hbar}\left\{PHQ\frac{1}{\mathcal{L}^{(0)}}QHP + \mathrm{H.c.}\right\}$$

$$+ \frac{\pi}{2\hbar}\{iPHQ\ \delta(\mathcal{L}^{(0)})\ QHP + \mathrm{H.c.}\} \tag{43}$$

stemming from δH.

In Σ those contributions that involve $\delta(\mathcal{L}^{(0)})$ may be omitted as long as the P and Q sectors of Hilbert space are associated with clearly distinct energies. Our result for the effective Hamiltonian, which we have obtained within the general framework of transport theory, is very similar to the *Bloch-Feshbach formula* known in the theory of nuclear dynamics [21] or to Anderson's *poor man's scaling* [14]. Below we wish to demonstrate how this result can be utilized to derive renormalization-group equations for a variety of physical systems.

V. RENORMALIZATION GROUP FOR INTERACTING QUANTUM GASES

A. Hamiltonian and ground state

As an illustration of the above general result we shall investigate the low-temperature properties of interacting quantum gases, i.e., the effective dynamics of low-energy excitations above their many-particle ground state. We assume the microscopic dynamics of the gas to be governed by a Hamiltonian of the form

$$H = H_{\mathrm{kin}} + V_{2\to 2}$$

$$= \sum_k \epsilon_k :a_k^\dagger a_k: + \frac{1}{4}\sum_{i,j,k,l}\langle lk|V|ji\rangle_\pm :a_l^\dagger a_k^\dagger a_j a_i: \ , \tag{44}$$

with kinetic energy H_{kin} and a two-body interaction $V_{2\to 2}$. The single-particle energies ϵ_k include the chemical potential. Annihilation and creation operators obey $[a_i, a_j^\dagger]_\mp = \delta_{ij}$ for bosons (upper sign) or fermions (lower sign), respectively.

Each term in the Hamiltonian is normal ordered ($: \ :$) with respect to the noninteracting many-particle ground state. For bosons this ground state has all particles in the lowest-energy, zero-momentum single-particle mode, while for fermions it consists of a filled Fermi sea with all momentum modes occupied up to some Fermi momentum K_F. (For simplicity, the Fermi surface will be taken to be spherical.) The explicit normal ordering of the Hamiltonian is redundant in the bosonic case. In the fermionic case, on the other hand, it means shuffling all operators that annihilate the fermionic vacuum (a_i for states above the Fermi surface, $a_i^\dagger$ for states below the Fermi surface) to the right and all others ($a_k^\dagger$ for states above, a_k for states below the Fermi surface) to the left, thereby changing sign depending on the degree of the permutation.

We will assume that, at least to a good approximation, the essential features of the ground state survive even in the presence of interaction. More specifically, we will assume that in the case of interacting bosons the ground state still has most particles in modes with zero, or at least very small, momentum and that in the case of interacting fermions there still exists a well-defined Fermi surface. Low-energy excitations then correspond to the promotion of bosons from small to some slightly higher momentum or of fermions from just below the Fermi surface to just above it [22]. At low temperature the regions of interest in momentum space are therefore the vicinity of the origin (bosons) or the vicinity of the Fermi surface (fermions), respectively.

B. Mode elimination

We now wish to devise a systematic procedure for focusing onto these regions of interest. To this end we consider effective theories (i) in the bosonic case for modes within a sphere around the origin, of radius Λ, and (ii) in the fermionic case for modes within a shell inclosing the Fermi surface, of mean radius K_F and thickness 2Λ (where $\Lambda \ll K_F$). Whereas in the limit of large Λ one recovers the original, full theory, the opposite limit $\Lambda \to 0$ yields the desired low-energy effective theory. In order to interpolate between these two limits we proceed in infinitesimal steps. We lower the cutoff from some given $\Lambda(s)$ to

$$\Lambda(s+\Delta s) := \exp(-\Delta s)\Lambda(s), \quad \Delta s \geqslant 0, \tag{45}$$

with Δs infinitesimal, thereby discarding from the theory momentum modes pertaining to an infinitesimal shell (in the fermionic case, two shells) of thickness $\Delta\Lambda = \Lambda(s)\,\Delta s$. We determine the effective dynamics of the remaining modes, then eliminate the next infinitesimal shell, again determine the effective dynamics of the remaining modes, and so on. After each infinitesimal step we obtain a new effective Hamiltonian with slightly modified coupling constants. These may also include couplings that had not been present in the original theory: In fact, the mode elimination procedure will typically generate an infinite number of such couplings. But in many cases only a few coupling constants will change appreciably and thus suffice to study the physical system at hand. How these coupling constants evolve as the flow parameter s increases and hence the cutoff $\Lambda(s)$ approaches zero can then be described by a small set of coupled differential equations. Modulo trivial scaling, these are the *renormalization-group equations* of the theory.

At a given cutoff Λ the many-particle Hilbert space (Fock space) for bosons is spanned by the particle-free vacuum $|0_b\rangle$ and all n-particle states ($n=1,\ldots,\infty$)

$$|\mathbf{k}_1 \cdots \mathbf{k}_n\rangle \propto \prod_{i=1}^{n} a^\dagger(\mathbf{k}_i)|0_b\rangle, \quad |\mathbf{k}_i| \leqslant \Lambda, \tag{46}$$

where the $\{\mathbf{k}_i\}$ denote the particle momenta and $\{a^\dagger(\mathbf{k}_i)\}$ the associated bosonic creation operators.

The fermion Fock space, on the other hand, is spanned by the filled Fermi sea (fermionic vacuum) $|0_f\rangle$ and all its excitations which have particles above the Fermi surface and/or missing particles (''holes'') below it, all within a shell of thickness 2Λ. In order to cast this into a mathematical formulation it is convenient to change coordinates, from the true particle momenta $\{\mathbf{K}_i\}$ to little (''quasiparticle'') momenta

$$\mathbf{k}_i := (|\mathbf{K}_i| - K_F)\ \hat{\mathbf{K}}_i \tag{47}$$

and additional discrete labels

$$\sigma_i := \mathrm{sgn}(|\mathbf{K}_i| - K_F). \tag{48}$$

This coordinate transformation $\mathbf{K} \to (\mathbf{k}, \sigma)$ is invertible except for modes that lie exactly on the Fermi surface. States above the Fermi surface are labeled $\sigma = 1$, while those below are labeled $\sigma = -1$. The allowed excitations in fermion Fock space then have the form ($n=1,\ldots,\infty$)

$$|\mathbf{k}_1^{\pm} \cdots \mathbf{k}_n^{\pm}\rangle \propto \prod_{i=1}^{n} [\,\theta(\sigma_i)a^\dagger(\mathbf{k}_i,\sigma_i) + \theta(-\sigma_i)a(-\mathbf{k}_i,\sigma_i)]|0_f\rangle\ , \quad |\mathbf{k}_i| \leqslant \Lambda, \tag{49}$$

where the $\{\mathbf{k}_i^{\pm}\}$ denote the momenta of particles $(+)$ or holes $(-)$, respectively, and $\{a^\dagger\}$ and $\{a\}$ the associated fermionic creation and annihilation operators, respectively. For simplicity, we have omitted any spin quantum numbers.

It is now obvious which form the projection operator will have that is associated with the infinitesimal cutoff reduction (45): If applied to any of the excitations (46) or (49) it will simply multiply the respective state by a product of θ functions $\Pi_i \theta(\Lambda - e^{\Delta s}|\mathbf{k}_i|)$ to enforce the new cutoff constraint.

C. Modification of the two-body interaction

Each mode elimination will yield an effective Hamiltonian that will generally contain a slightly altered mass, chemical potential, two-body interaction, etc., and possibly new interactions such as an effective three-body interaction. Here we shall restrict our attention to the modification of the two-body interaction. This modification is entirely due to the extra term Σ [Eq. (43)] in the effective Hamiltonian,

$$\Sigma = -\frac{1}{32\hbar}\Bigg\{ \sum_{a,b,c,d} \sum_{i,j,k,l} \langle lk|V|ji\rangle_{\pm}\langle dc|V|ba\rangle_{\pm} \times P{:}a_l^\dagger a_k^\dagger a_j a_i{:}Q\,\frac{1}{\mathcal{L}^{(0)}}\,Q{:}a_d^\dagger a_c^\dagger a_b a_a{:}P + \mathrm{H.c.}\Bigg\}\ , \tag{50}$$

where to the given order in perturbation theory $\mathcal{L}^{(0)}$ just coincides with $\mathcal{L}_{\mathrm{kin}}$. The two projectors P at both ends of the operator product ensure that all external momenta lie below the new, reduced cutoff, whereas the projectors Q in the center force at least one internal momentum to lie in that infinitesimal shell which has just been eliminated. Therefore, at least one pair of field operators must pertain to the eliminated Q modes and hence be contracted. The product of the remaining six field operators can then be rearranged with the help of Wick's theorem to yield a decomposition

$$\Sigma = \Sigma_6 + \Sigma_4 + \Sigma_2 + \Sigma_0, \tag{51}$$

each Σ_n being a normal-ordered product of n field operators. The various terms shift the ground-state energy ($n=0$); modify the mass, the chemical potential, or more generally the form of the single-particle dispersion relation ($n=2$); modify the two-body interaction ($n=4$); and generate a new effective three-body interaction ($n=6$).

As we want to focus on the modification of the two-body interaction, we consider only the term with $n=4$. Neglecting the energy of the external modes we find for bosons ($\hbar = 1$)

$$\Delta\langle lk|V|ji\rangle_+ = -\Delta\left[\sum_{a,b}\frac{1}{2(\epsilon_a+\epsilon_b)}\langle lk|V|ba\rangle_+\langle ab|V|ji\rangle_+\right] \tag{52}$$

and for fermions

$$\Delta\langle lk|V|ji\rangle_- = \Delta^{(\mathrm{ZS})}_{lk|ji}+\Delta^{(\mathrm{ZS}')}_{lk|ji}+\Delta^{(\mathrm{BCS})}_{lk|ji}, \tag{53}$$

with

$$\Delta^{(\mathrm{ZS})}_{lk|ji} = -\Delta\left[\sum_{a,b}\frac{\theta(\sigma_a)\theta(-\sigma_b)-\theta(-\sigma_a)\theta(\sigma_b)}{\epsilon_a-\epsilon_b}\times\langle la|V|bi\rangle_-\langle bk|V|ja\rangle_-\right], \tag{54}$$

its cross term

$$\Delta^{(\mathrm{ZS}')}_{lk|ji} = -\Delta^{(\mathrm{ZS})}_{kl|ji}, \tag{55}$$

and

$$\Delta^{(\mathrm{BCS})}_{lk|ji} = -\Delta\left[\sum_{a,b}\frac{\theta(\sigma_a)\theta(\sigma_b)-\theta(-\sigma_a)\theta(-\sigma_b)}{2(\epsilon_a+\epsilon_b)}\times\langle lk|V|ba\rangle_-\langle ab|V|ji\rangle_-\right]. \tag{56}$$

The Δ in front of the sums signifies that at least one of the internal modes (a,b) must lie in the eliminated shell. In the bosonic case the modification of the two-body interaction can be associated with a one-loop ''ladder'' diagram. In the fermionic case, on the other hand, there are three distinct contributions which, with hindsight, may be identified with ''zero sound'' (ZS,ZS$'$) and BCS diagrams [11]. The ZS contribution and its cross term ZS$'$ account for particle-hole excitations ($\sigma_a=\pm1,\sigma_b=\mp1$), while the BCS term describes two-particle ($\sigma_a=\sigma_b=+1$) or two-hole ($\sigma_a=\sigma_b=-1$) excitations.

The above formulas serve as the starting point for the investigation of a variety of specific physical systems. With their help one can derive such diverse results as the one-loop renormalization-group equation for an interacting Bose gas, the one-loop β function of ϕ^4 theory, the screening of fermion-fermion interactions, or the BCS instability. Details of these applications are presented in the following section.

VI. EXAMPLES

A. Bosons with point interaction

Our first example pertains to spinless bosons with a point interaction (δ function potential in real space)

$$\langle lk|V|ji\rangle_+ = \frac{2U}{\Omega}\,\delta_{\mathbf{k}_i+\mathbf{k}_j,\mathbf{k}_k+\mathbf{k}_l}, \tag{57}$$

with the Kronecker symbol enforcing momentum conservation, Ω being the spatial volume, and U the coupling constant. Provided the magnitude of the external momenta $\mathbf{k}_i$ and $\mathbf{k}_j$ is negligible compared to the cutoff Λ, momentum conservation implies that the internal modes a,b must *both* lie in the eliminated shell, and hence $\epsilon_a=\epsilon_b=\epsilon_\Lambda$. Application of the general formula (52) then yields

$$\Delta U = -\frac{U^2}{2\Omega\epsilon_\Lambda}\Delta\left[\sum_{|\mathbf{k}_a|,|\mathbf{k}_b|\leqslant\Lambda}\delta_{\mathbf{k}_i+\mathbf{k}_j,\mathbf{k}_a+\mathbf{k}_b}\right], \tag{58}$$

where the sum

$$\Delta\left[\sum_{|\mathbf{k}_a|,|\mathbf{k}_b|\leqslant\Lambda}\delta_{\mathbf{k}_i+\mathbf{k}_j,\mathbf{k}_a+\mathbf{k}_b}\right]\approx\Delta\left[\sum_{|\mathbf{k}_a|,|\mathbf{k}_b|\leqslant\Lambda}\delta_{\mathbf{k}_b,-\mathbf{k}_a}\right] = \sum_{|\mathbf{k}_a|\in[\Lambda-\Delta\Lambda,\Lambda]}1 \tag{59}$$

simply counts the number of eliminated states. For a spherical cut in momentum space this number of states is given by

$$\sum_{|\mathbf{k}_a|\in[\Lambda-\Delta\Lambda,\Lambda]}1 = \rho(\epsilon_\Lambda)\,\frac{d\epsilon_\Lambda}{d\Lambda}\,\Delta\Lambda, \tag{60}$$

with $\rho(\epsilon_\Lambda)$ denoting the density of states at the cutoff. With $\Delta\Lambda=\Lambda\,\Delta s$ we thus obtain the flow equation

$$\Delta U = -\frac{d\ln\epsilon_\Lambda}{d\ln\Lambda}\,\frac{\rho(\epsilon_\Lambda)}{2\Omega}\,U^2\Delta s. \tag{61}$$

For a dilute gas of nonrelativistic bosons in three spatial dimensions, with mass m, dispersion relation $\epsilon_\Lambda=\Lambda^2/2m$, and density of states $\rho(\epsilon_\Lambda)=\Omega m\Lambda/2\pi^2$ the flow equation reduces to

$$\Delta U = -\frac{m\Lambda}{2\pi^2}U^2\Delta s. \tag{62}$$

By its very definition the sequence of effective theories retains complete information about the system's low-energy dynamics. Observables pertaining to this low-energy dynamics are therefore unaffected by the successive mode elimination and hence independent of s. For example, the scattering length [23]

$$a = \frac{m}{4\pi}\left[U(s)-U(s)^2\int_{|\mathbf{p}|\leqslant\Lambda(s)}\frac{d^3p}{(2\pi)^3}\,\frac{m}{\mathbf{p}^2}\right] \tag{63}$$

stays constant under the flow (62), as the s dependence of the parameters U and Λ just cancels out (up to third-order corrections).

B. The link to ϕ^4 theory

There is an interesting relationship between the result (61) and the one-loop β function for real ϕ^4 theory. The ϕ^4 Hamiltonian describes the dynamics of coupled anharmonic oscillators. It reads, in three spatial dimensions, $H=H^{(0)}+V$ with kinetic energy

$$H^{(0)} = \frac{1}{2}\int d^3x{:}[\,\pi(x)^2+|\nabla\phi(x)|^2+m^2\phi(x)^2]{:} = \sum_{\mathbf{k}}\epsilon_{\mathbf{k}}\,a^\dagger_{\mathbf{k}}a_{\mathbf{k}} \tag{64}$$

and interaction

$$V=\frac{g}{4!}\int d^3x\ \phi(x)^4$$

$$=\frac{g}{4!\Omega}\sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4}\prod_{\alpha=1}^{4}\frac{1}{\sqrt{2\epsilon_{\mathbf{k}_\alpha}}}(a_{\mathbf{k}_\alpha}+a^\dagger_{-\mathbf{k}_\alpha})\delta_{\Sigma\mathbf{k}_i,0}. \tag{65}$$

Here m denotes the mass, Ω the spatial volume, g the coupling constant, and $\epsilon_{\mathbf{k}}$ the single-particle energy

$$\epsilon_{\mathbf{k}}=\sqrt{\mathbf{k}^2+m^2}. \tag{66}$$

The field ϕ and its conjugate momentum π are time-independent (Schrödinger picture) operators that satisfy the commutation relations for bosons, and $a,a^\dagger$ are the associated annihilation and creation operators, respectively. While the kinetic part of the Hamiltonian is normal ordered $(::)$, the interaction is not.

When expressed in terms of annihilation and creation operators the Hamiltonian takes on a form that is very similar to that of the quantum gas Hamiltonian (44). More precisely, the ϕ^4 Hamiltonian *contains* a Bose gas Hamiltonian with two-body interaction matrix element

$$\langle lk|V|ji\rangle_+=\binom{4}{2}\frac{g}{4!\Omega\sqrt{\epsilon_i\epsilon_j\epsilon_k\epsilon_l}}\delta_{\mathbf{k}_i+\mathbf{k}_j,\mathbf{k}_k+\mathbf{k}_l}. \tag{67}$$

The derivation of a flow equation for g can now proceed in the same vein as that for U, again starting from Eq. (52). Now, however, apart from $2\to2$ particle scattering, the ϕ^4 Hamiltonian with its additional interactions $a^\dagger a^\dagger a^\dagger a$, $a^\dagger a^\dagger a^\dagger a^\dagger$, etc., also permits $2\to4$ and $2\to6$ scattering. Therefore, in Eq. (52) the intermediate state may be not just $|ab\rangle$, but also $|abik\rangle$, $|abil\rangle$, $|abjk\rangle$, $|abjl\rangle$, or $|abijkl\rangle$. As long as the magnitude of the external momenta is negligible compared to the cutoff, it is in all six cases $\epsilon_a=\epsilon_b=\epsilon_\Lambda$ and

$$\langle lk|V|\cdots\rangle_+\langle\cdots|V|ji\rangle_+=\frac{g}{4\Omega\epsilon_\Lambda^2}\langle lk|V|ji\rangle_+\,\delta_{\mathbf{k}_b,-\mathbf{k}_a}\ . \tag{68}$$

Hence in order to account for the larger set of allowed intermediate states we merely have to introduce an extra factor 6 and obtain thus

$$\Delta g=-\frac{d\ln\epsilon_\Lambda}{d\ln\Lambda}\,\frac{3\rho(\epsilon_\Lambda)}{8\Omega\epsilon_\Lambda^2}\,g^2\Delta s. \tag{69}$$

For $\Lambda\gg m$ it is $\epsilon_\Lambda=\Lambda$, $\rho(\epsilon_\Lambda)=\Omega\epsilon_\Lambda^2/2\pi^2$ and the flow equation reduces to

$$\Delta g=-\frac{3g^2}{16\pi^2}\Delta s, \tag{70}$$

in agreement with the well-known one-loop result for the β function of ϕ^4 theory [24,25].

C. Screening of fermion-fermion interactions

We consider nonrelativistic fermions in spatial dimension d $(d\geqslant2)$ that interact through a two-body interaction

$$\langle lk|V|ji\rangle_-=[V(\mathbf{q})\delta_{s_is_i}\delta_{s_ks_j}-V(\mathbf{q}')\delta_{s_ks_i}\delta_{s_ls_j}]\delta_{\mathbf{K}_i+\mathbf{K}_j,\mathbf{K}_k+\mathbf{K}_l}, \tag{71}$$

duly antisymmetrized to account for Fermi statistics, and with $\{s_\alpha\}$ denoting the spin quantum numbers and $\mathbf{q},\mathbf{q}'$ the respective momentum transfers

$$\mathbf{q}:=\mathbf{K}_l-\mathbf{K}_i=\mathbf{K}_j-\mathbf{K}_k,$$

$$\mathbf{q}':=\mathbf{K}_k-\mathbf{K}_i=\mathbf{K}_j-\mathbf{K}_l. \tag{72}$$

We investigate scattering processes for which

$$0<|\mathbf{q}|,\Lambda\ll|\mathbf{q}'|,|\mathbf{K}_i+\mathbf{K}_j|,K_F. \tag{73}$$

In this regime only the ZS contribution (54) can significantly modify the two-body interaction; its cross term ZS′ [Eq. (55)] as well as the BCS contribution (56) are suppressed by a factor Λ/K_F. This can be seen directly from the geometry of the Fermi surface. The three constraints on the intermediate state—(i) both $\mathbf{K}_a$ and $\mathbf{K}_b$ lie in the cutoff shell of thickness 2Λ; (ii) more stringently, one of them lie in the infinitesimal shell to be eliminated; and (iii) $\mathbf{K}_a-\mathbf{K}_b=-\mathbf{q}'$ (ZS′) or $\mathbf{K}_a+\mathbf{K}_b=\mathbf{K}_i+\mathbf{K}_j$ (BCS), respectively—reduce the momentum space volume available to the internal momentum $\mathbf{K}_a$ to $O(K_F^{d-2}\Lambda\Delta\Lambda)$. In contrast, for $|\mathbf{q}|\sim\Lambda$ the ZS contribution with its condition $\mathbf{K}_a-\mathbf{K}_b=-\mathbf{q}$ allows a momentum space volume of the order $K_F^{d-1}\Delta\Lambda$.

To evaluate the ZS contribution at some given momentum transfer $\mathbf{q}$, we first define the angle ϑ between $-\mathbf{q}$ and the internal momentum $\mathbf{K}_a$,

$$\cos\vartheta\equiv z:=-\frac{\mathbf{q}\cdot\mathbf{K}_a}{|\mathbf{q}||\mathbf{K}_a|}, \tag{74}$$

change coordinates from original $(\mathbf{K})$ to little $(\mathbf{k})$ momenta, and write, up to corrections of order $|\mathbf{q}|/K_F$,

$$\epsilon_a-\epsilon_b=v_F(|\mathbf{K}_a|-|\mathbf{K}_b|)=v_F(\sigma_a|\mathbf{k}_a|-\sigma_b|\mathbf{k}_b|)$$

$$=v_F|\mathbf{q}|z, \tag{75}$$

with v_F denoting the Fermi velocity. Next we note that the term with $\theta(\sigma_a)\theta(-\sigma_b)$ and the term with $\theta(-\sigma_a)\theta(\sigma_b)$ yield identical contributions; therefore, it suffices to consider only the first term and then multiply it by 2. Finally, assuming that in the interaction matrix element (71) the cross term is negligible,

$$|V(\mathbf{q}')|\ll|V(\mathbf{q})|, \tag{76}$$

the two matrix elements in Eq. (54) can simply be replaced by $V(\mathbf{q})^2$ modulo Kronecker symbols for spin and momentum conservation. By virtue of these Kronecker symbols one of the two summations over internal modes collapses trivially, leaving

$$\Delta V(\mathbf{q})=-\frac{2}{v_F}\Delta\Bigg[\sum_a\ \theta(\Lambda-|\mathbf{k}_a|)\,\theta(|\mathbf{k}_a|-|\mathbf{q}|z+\Lambda)$$

$$\times\frac{\theta(\sigma_a)\,\theta(|\mathbf{q}|z-|\mathbf{k}_a|)}{|\mathbf{q}|z}\Bigg]V(\mathbf{q})^2. \tag{77}$$

Here the first two θ functions explicitly enforce the sharp

cutoff constraint for both $\mathbf{k}_a$ and $\mathbf{k}_b$ ($|\mathbf{k}_a|,|\mathbf{k}_b| \leqslant \Lambda$), while the latter two θ functions enforce $\sigma_a = 1$ and $\sigma_b = -1$, respectively. Under these constraints it is always $\vartheta \in [0,\pi/2)$ and hence $z > 0$.

The above equation can be immediately integrated from cutoff $\Lambda \gg |\mathbf{q}|,|\mathbf{k}_a|$ (symbolically, $\Lambda \to \infty$) down to $\Lambda \ll |\mathbf{q}|,|\mathbf{k}_a|$ (symbolically, $\Lambda \to 0$), to yield the total modification of the two-body interaction

$$\frac{1}{V_{\text{eff}}(\mathbf{q})} - \frac{1}{V_{\text{bare}}(\mathbf{q})} = \frac{2}{v_F} \sum_a \left. \frac{\theta(\Lambda - |\mathbf{k}_a|)\,\theta(|\mathbf{k}_a| - |\mathbf{q}|z + \Lambda)\,\theta(\sigma_a)\,\theta(|\mathbf{q}|z - |\mathbf{k}_a|)}{|\mathbf{q}|z} \right|_0^\infty . \tag{78}$$

At the lower bound ($\Lambda \to 0$) the various conditions imposed by the θ functions cannot all be satisfied simultaneously and therefore the product of θ functions vanishes. At the upper bound ($\Lambda \to \infty$), on the other hand, the cutoff constraints imposed by the first two θ functions are trivially satisfied and thus can be omitted. In this case the sum over a is evaluated by turning it into two integrals, one over a radial variable such as $|\mathbf{k}_a|$ or ϵ_a, the other over the solid angle. At a given solid angle and hence given z, the fourth θ function restricts the radial integration to the range $|\mathbf{k}_a| \in [0,|\mathbf{q}|z]$ or, equivalently, $\epsilon_a \in [0,v_F|\mathbf{q}|z]$. This energy interval in turn corresponds to a number $(\rho(\epsilon_F)v_F|\mathbf{q}|z)$ of states, $\rho(\epsilon_F)$ being the density of states at the Fermi surface. (The density of states takes any spin degeneracy into account.) The integration over the solid angle is constrained to a semisphere, due to $\vartheta \in [0,\pi/2)$, and hence reduced by a factor 1/2 as compared to a full-sphere integration. Altogether we obtain

$$\frac{1}{V_{\text{eff}}(\mathbf{q})} - \frac{1}{V_{\text{bare}}(\mathbf{q})} = \frac{2}{v_F}\ \frac{1}{2}\ \rho(\epsilon_F) v_F |\mathbf{q}|z\ \frac{1}{|\mathbf{q}|z} = \rho(\epsilon_F) \tag{79}$$

and thus

$$V_{\text{eff}}(\mathbf{q}) = \left[\frac{1}{V_{\text{bare}}(\mathbf{q})} + \rho(\epsilon_F) \right]^{-1} . \tag{80}$$

This result describes the well-known screening of fermion-fermion interactions [26].

D. BCS instability

Our last example pertains to fermions with an attractive pairing interaction

$$\langle lk|V|ji\rangle_- = -V\ \delta_{\mathbf{K}_j,-\mathbf{K}_i}\delta_{\mathbf{K}_l,-\mathbf{K}_k}[\,\delta_{s_l s_i}\delta_{s_k s_j} - \delta_{s_k s_i}\delta_{s_l s_j}]\ , \tag{81}$$

which is the simplest form of BCS theory [27]. Due to the pairing condition $\mathbf{K}_j = -\mathbf{K}_i$, $\mathbf{K}_l = -\mathbf{K}_k$ it is impossible to satisfy in the ZS and ZS′ terms the requirement that at least one of the internal modes be in the eliminated shell. Hence only the BCS term (56) can modify the coupling constant. In the BCS term there are contributions with $\theta(\sigma_a)\theta(\sigma_b)$ and $\theta(-\sigma_a)\theta(-\sigma_b)$, respectively, which yield identical results; therefore, it suffices to consider only the first contribution and then multiply it by 2. The pairing condition implies $\epsilon_a = \epsilon_b = \epsilon_\Lambda$, which for modes in the upper ($\sigma = +1$) eliminated shell is given by $\epsilon_\Lambda = v_F\Lambda$. The eliminated shell itself covers an infinitesimal energy interval of width $(v_F\Lambda\Delta s)$, which in turn corresponds to a number $(\rho(\epsilon_F)v_F\Lambda\Delta s)$ of states. Of the two summations over internal modes one collapses trivially due to momentum and spin conservation, leaving

$$\begin{aligned} \Delta V &= \frac{V^2}{2v_F\Lambda}\Delta\left[\sum_a\ \theta(\sigma_a)\right] \\ &= \frac{V^2}{2v_F\Lambda}\ \rho(\epsilon_F)\ v_F\Lambda\Delta s = \frac{\rho(\epsilon_F)}{2}\ V^2\Delta s. \end{aligned} \tag{82}$$

From this flow equation for the BCS coupling $V(s)$ we immediately conclude that as long as the initial coupling $V(0)$ is positive, $V(s)$ diverges as $s \to \infty$. This indicates the occurrence of binding (''Cooper pairs'') at very low temperatures. Furthermore, we can again convince ourselves that the sequence of effective theories retains complete information about the system's low-energy dynamics: Low-energy observables such as the zero-temperature gap [27]

$$\Delta_0 = 2\Lambda(s)\exp\left[-\frac{2}{\rho(\epsilon_F)V(s)} \right] \tag{83}$$

do not depend on the flow parameter s and are thus unaffected by the successive mode elimination.

VII. CONCLUSION

We have succeeded in linking renormalization to transport theory. Our line of argument proceeded from the exact microscopic dynamics, via the Nakajima-Zwanzig projection technique, to a macroscopic transport equation for selected expectation values and then, via second-order perturbation theory, Markovian limit, choice of the Robertson projector, and a suitable rearrangement of terms in the transport equation, to the approximate effective Hamiltonian that governs the nondissipative part of the macroscopic dynamics. We investigated the ramifications of this result for the low-energy dynamics of interacting quantum gases: Contracting the set of selected expectation values by discarding iteratively those observables that pertain to short-wavelength excitations, we obtained a sequence of effective Hamiltonians that describe the dynamics on successively larger length scales. We then focused on the two-body interaction in these effective Hamiltonians and convinced ourselves, in several rather diverse applications, that it varies in accordance with

one-loop renormalization-group equations. We have thus substantiated our original claim that renormalization-group equations can be obtained within the Nakajima-Zwanzig projection approach and hence renormalization can be embedded into the general mathematical framework of transport theory.

There remain many open questions worth investigating. Clearly, a unified theoretical framework for dissipative transport and renormalization will have to prove its merits in new applications where the conventional approaches fail: for example, when time scales are no longer well separated and the Markovian limit ceases to be justified; or when, in the course of successive mode elimination, one starts to discard states with a finite population, thus introducing dissipation into the effective macroscopic dynamics. It is my hope that the present paper will help stimulate research efforts in these and related directions.

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